ISRAEL JOURNAL OF MATHEMATICS, Vol. 23, Nos. 3–4, 1976

ON EMBEDDING OF GROUP RINGS IN DIVISION RINGS

BY

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ABSTRACT

Let F be a free group, $N \lhd F$ and $V(N)$ be a verbal subgroup of N. For the group ring $R(\bar{F})$, where R is any field and $\bar{F} = F/V(N)$, the zero divisor problem of Kaplansky and the problem of embedding $R(\bar{F})$ in a division ring are investigated. It is proved, in particular, that $R(\bar{F})$ has no zero divisors and can be embedded in a division ring when F/N is finitely approximated and $N/V(N)$ is approximated by nilpotent groups without torsion.

Introduction

Let F be a free group, N a normal subgroup, and $V(N) \neq N$ a fully invariant subgroup of N.

Let us suppose that the factorgroup $\bar{N} = N/V(N)$ is a group without torsion. It is known that in this case the factorgroup $\bar{F} = F/V(N)$ is also a group without torsion (see, for example, [1]).

We shall investigate in this paper the group ring $R(\bar{F})$ of a group $\bar{F}$ over any field R and we shall get some sufficient conditions for $R(\bar{F})$ to be embedded in a division ring.

THEOREM 1. *If $\bar{N} = N/V(N)$ is a nilpotent group without torsion and $G = F/N$ has a subnormal series (possibly of transfinite length)*

$$G \cdots \rhd H_\alpha \cdots \rhd H_2 \rhd H_1 \rhd E, \tag{1}$$

where all factors $H_{\alpha+1}/H_\alpha$ are locally finite or locally solvable groups, then the group ring $R(\bar{F})$ has no divisors of zero and has division ring of quotients.

We shall say that a group G is a group of class A if as above G has a subnormal series (1) where all factors $H_{\alpha+1}/H_\alpha$ are locally solvable or locally finite groups; the series (1) we shall call an A-series.

Received June 16, 1975

The next theorem gives a sufficient condition for the embedding of $R(\bar{F})$ in a division ring for a wider class of groups $\bar{F}$, than that considered in Theorem 1. However, in general, in this case $R(\bar{F})$ has no full division ring of quotients, as can be seen from the example of ring $R(F)$, where F is a free group.

THEOREM 2. *Let F be a free group and $N \lhd F$. If F/N is approximated by groups from the class A and $V(N) \neq N$ is a fully invariant subgroup of N, such that $\bar{N} = N/V(N)$ is approximated by nilpotent groups without torsion, then the group ring $R(F)$ of the group $\bar{F} = F/V(N)$ has no divisors of zero and can be embedded in a division ring.*

If G is any group, we will denote by $\Gamma_n(G)$ the nth term of the lower central series. Thus, $\Gamma_1(G) = G, \Gamma_2(G)$ is the commutator subgroup $[G, G]$ of G, and $\Gamma_n(G) = [\Gamma_{n-1}(G), G]$. By $\Gamma_{n_1, n_2, \cdots, n_k}(G)$ we will denote the term of an iterated polycentral lower central series, that corresponds to the sequence of natural numbers $n_1, n_2, \cdots, n_k$.

We can get from Theorem 2 the next consequence (see Corollary 2).

If G/N is a finitely approximated group, then the group ring $R(\bar{F})$ of the group $\bar{F} = F/\Gamma_{n_1, n_2, \cdots, n_k}(F)$ has no divisors of zero and can be embedded in a division ring.

Smirnov has proved the absence of divisors of zero in the case when $V(N)$ is the commutator subgroup of N and G is an RN-group (See [12]). All other results on the zero divisor problem in group ring are contained in [10] and [8].

I am grateful to the reviewer for his helpful remarks.

We shall state now a few lemmas that we will need for the proof of Theorem 1.

First of all, we recall that a ring Q has a (right) ring of quotients if and only if it satisfies an Ore condition: for any two elements $q_1, q_2 \in Q$, where q_2 is regular, there exist elements $r_1, r_2 \in Q$, such that

$$q_1 r_1 = q_2 r_2 . \tag{2}$$

A ring without divisors of zero and with condition (2) *will be called an Ore ring*; as is well-known, such a ring has a full division ring of quotients.

The conclusions of the following lemma are well-known.

LEMMA 1. *Let G be any group and $R(G)$ its group ring over the field R. Then*

(a) *If $R(G)$ is an Ore ring and G_1 is a subgroup of G, then $R(G_1)$ is an Ore ring too.*

(b) *If for every finitely generated subgroup $K \subseteq G$ the group ring $R(K)$ is an Ore ring, then $R(G)$ is an Ore ring too.*

(c) *Let $G \rhd H$ and $R(H)$ be an Ore ring. There exists a ring $R_H(G)$ of the quotients of $R(G)$ over $R(H)$ and $R(G)$ is an Ore ring if and only if $R_H(G)$ is an Ore ring.*

(d) *Let $G \rhd H$ and $R(H)$ be an Ore ring. If G/H is an infinite cyclic group then $R(G)$ is an Ore ring too.*

PROOF.

(a) Any element $z \in R(G)$ can be written uniquely in the form $z = \Sigma_{i=1}^{n} \lambda_i g_i$, where $\lambda_i \in R(G_1), i = 1, 2, \cdots, n$ and $g_1 = 1, g_2 \cdots g_n$ are some representatives of right cosets of G over G_1.

Let α and β be any elements from $R(G_1)$. There exist $x = \Sigma \alpha_i g_i$ and $y = \Sigma \beta_i g_i$ in $R(G)$ such that

$$\alpha x = \beta y \cdot \tag{3}$$

The equation (3) gives us immediately:

$$\alpha\alpha_1 = \beta\beta_1, \tag{3'}$$

which means that the condition (2) holds in $R(G_1)$. The absence of divisors of zero in $R(G_1)$ follows immediately. Hence, $R(G_1)$ is an Ore ring.

(b) Follows immediately from condition (2).

(c) Let D be the full ring of (right) quotients of $R(H)$. Thus D is a division subring of $R_H(G)$ and for any element $\lambda \in D$ we have $\lambda = \alpha\beta^{-1}, \alpha, \beta \in R(H)$. The statement now follows easily from the existence of the representations, for every $x \in R_H(G)$,

$$x = \sum_{i=1}^{n} \lambda_i g_i \tag{4}$$

and

$$x = \sum_{i=1}^{n} g_i \lambda'_i, \tag{4'}$$

where $g_1 = 1, g_2 \cdots, g_n$ are some representatives of cosets G over H and $\lambda_i, \lambda'_i \in D, i = 1, 2, \cdots, n$.

d) Let gH be the generator of the factorgroup G/H. Then from (4) we get that there exists a unique representation for every $x \in R_H(G)$

(4″) $$x = \sum_{i=-k}^{n} \lambda_i g^i.$$

It is easy to see now that $R_H(G)$ has no divisors of zero; moreover, an algorithm of division can be determined in $R_H(G)$ as in the case of the ring of noncommutative polynomials $Q(t, S)$ and, as in the latter case, we obtain that $R_H(G)$ is a principal ideal ring (see [6], chap. 3). Thus, $R_H(G)$ is an Ore ring and from c) follows now that $R(G)$ is also an Ore ring.

LEMMA 2. *Let F be a free group, $N \lhd F$ and $G = F/N$ be a solvable group. If $H \subseteq \Gamma_2(N)$ is a normal subgroup of F such that the group ring $R(\bar{N})$ of the group $\bar{N} = N/H$ is an Ore ring, then the group ring $R(\bar{F})$ of the group $\bar{F} = F/N$ is an Ore ring too.*

PROOF. Let $F = F^0 \supseteq F' \supseteq \cdots F^{(m)} \supseteq \cdots$ be the derived series of F and n be the first number such that $F^{(n)} \subseteq N$. We shall prove the lemma by induction on n.

If $n = 0$ the statement is trivial. So, we can suppose that $n \geqq 1$ and the lemma is proved for such pairs F and N, that $F^{(s)} \subseteq N, s \leqq n-1$.

We have now for the group $\Gamma_2(F)N$

(5) $$\Gamma_2(F)N \rhd N \supseteq \Gamma_2(N) \supseteq H,$$

where

$$H \lhd \Gamma_2(N) \text{ and } (\Gamma_2(F)N)^{(n-1)} \subseteq N.$$

It follows now from the supposition of induction that $R(\overline{\Gamma_2(F)N})$ is an Ore ring. Lemma 1(a) shows that $R(\overline{\Gamma_2(F)})$ is an Ore ring too. We also have

(5′) $$\Gamma_2(F) \rhd \Gamma_2(F) \cap N \supseteq \Gamma_2(N) \supseteq H.$$

Hence, $\bar{F}/\overline{\Gamma_2(F)}$ is a free abelian group.

If now the subgroup $\bar{K} \subseteq \bar{F}$ is generated by finite number of elements $\bar{g}_1, \bar{g}_2, \cdots, \bar{g}_k$, we have that $\bar{K} \subseteq K^* = \{\overline{\Gamma_2(F)}, \bar{g}_1, \bar{g}_2, \cdots, \bar{g}_k\}$. The factorgroup $K^*/\overline{\Gamma_2(F)}$ is a free abelian group of some finite rank r and hence K^* can be obtained from $\overline{\Gamma_2(F)}$ by r consecutive extensions by infinite cyclic groups. We have now from Lemma 1 that $R(K^*)$ and $R(\bar{K})$ are Ore rings.

As $\bar{K}$ is an arbitrary finitely generated subgroup of $\bar{F}$, we obtain too that $R(\bar{F})$ is an Ore ring. The lemma is proved.

We now need the next lemma about the groups of the class A.

LEMMA 3. *The class of groups A is closed under* a) *subgroups* b) *homomorphic images* c) *direct products.*

The proof is evident.

COROLLARY. *Let G be any group, H_i, $i = 1, 2, \cdots, k$ normal subgroups of G such that the factorgroups $\bar{G}_i = G/H_i$ are groups of the class A. Then the group $\bar{G} = G/H$, where $H = \bigcap_{i \in I} H_i$ is a group of the class A also.*

PROPOSITION 1. *Let F be a free group, $N \lhd F$, and $G = F/N$ be a group of the class A. If $H \subseteq \Gamma_2(N)$ is a normal subgroup of F such that the group ring $R(\bar{N})$ of the group $\bar{N} = N/H$ is an Ore ring, then the group ring $R(\bar{F})$ of the group $\bar{F} = F/H$ is an Ore ring also.*

PROOF. We will prove the proposition by induction on the length of the series (1). Thus, we shall suppose that there exists in the group $G = F/N$ a series with length α and the statement is proved for pairs F and N such that the length of the series (1) for the group F/N is less than α.

Let us consider the arbitrary subgroup $\bar{K} \subseteq \bar{F}$, which is generated by a finite number k of elements $\bar{f}_1, \bar{f}_2, \cdots, \bar{f}_k$; let f_i be some representative of class $\bar{f}_i$, $i = 1, 2, \cdots, k$ and $S = \mathrm{gp}\{N, f_1, f_2, \cdots, f_k\}$. We see that $S \rhd N \supseteq H$ and $\bar{S} = S/H \supseteq \bar{K}$.

As in the proof of Lemma 2 it is enough to prove that $R(\bar{S})$ is an Ore ring.

The factorgroup $T = S/N$ is a subgroup of G and thus there exists in T a series of type (1) with length $\beta \leqq \alpha$; as T is finitely generated we can suppose that $\beta - 1$ exists. So,

$$T = H_\beta \rhd H_{\beta-1}, \tag{6}$$

where the factorgroup $Q = H_\beta/H_{\beta-1} = T/H_{\beta-1}$ is either a finitely generated solvable group or a finite group.

Let S_1 be the inverse image of $H_{\beta-1}$ according to the natural homomorphism $\varphi: S \xrightarrow{\varphi} S/N = T$. Hence,

$$S \rhd S_1 \rhd N \rhd H, \tag{7}$$

and

$$S/S_1 \simeq Q, \quad S_1/N \simeq H_{\beta-1}.$$

There exists in the group $T = S_1/N$ an A-series with length $\leqq \beta - 1 < \alpha$. Hence, the ring $R(\bar{S}_1)$ of the group $\bar{S}_1 = S_1/H$ is an Ore ring.

We shall consider now two possibilities.

1) The factorgroup $S/S_1 \simeq Q$ is a solvable group.

In this case we obtain immediately from lemma 2 that $R(\bar{S})$ is an Ore ring.

2) The group Q is finite.

Let the order of Q be n, p be a prime dividing n, Q_p be a Sylow p-subgroup of Q and S_p be the inverse image of Q_p according to the natural homomorphism $\psi: S \xrightarrow{\psi} S/S_1 \simeq Q$. Hence,

$$S \supseteq S_p \rhd S_1 \rhd N \rhd H, \tag{8}$$

and $S_p/S_1 \simeq Q_p$; i.e. S_p/S_1 is a solvable group. As in the case 1) we now have that $R(\bar{S}_p)$ is an Ore ring.

Let us consider now the ring $D = R_{\bar{S}_1}(\bar{S})$ of quotients of $R(\bar{S})$ over $R(\bar{S}_1)$. We shall prove that D is a division ring and thus, according to Lemma 1c), $R(\bar{S})$ is an Ore ring.

Let D_1 be the whole division ring of quotients of $R(\bar{S}_1)$, $D_P = R_{\bar{S}_1}(\bar{S}_P)$ be the ring of quotients of $R(\bar{S}_P)$ over $R(\bar{S}_1)$. Since $R(S_P)$ is an Ore ring then from Lemma 1c) it follows that D_P is an Ore ring too.

It is easy to see now that the left dimension $\dim_l(D: D_1) = n$ and $\dim_l(D_p: D_1) = p^k$, where p^k is the highest degree of p that divides n. (p^k is the order of the Sylow p-subgroup Q_p.) Thus, D_p is an Ore ring which is finite dimensional over a division ring D_1. It is well-known that a ring without zero divisors, which is finite dimensional over a division ring, is itself a division ring.

Let M be any left nonzero ideal of D. Then

$$\dim_l(M:D_1) = \dim_l(M:D_p) \cdot \dim_l(D_p: D_1) \equiv 0 \pmod{p^K}. \tag{9}$$

As p is any prime divisor of n, we may obtain from (9) immediately

$$\dim_l(M: D_1) \equiv 0 \pmod{n}, \tag{10}$$

and hence, as $M \neq 0$,

$$\dim_l(M:D_1) = n, \text{ i.e. } M = D.$$

We have obtained that D has no nontrivial left ideals. Thus D is a division ring. So, as mentioned above, $R(\bar{S})$ is the Ore ring and the proposition is proved.

THEOREM 1. *Let F be a free group, $N \lhd F$ and $G = F/N$ be a group of the class A. If $V(N) \neq N$ is a fully invariant subgroup of N, such that the factorgroup $\bar{N} = N/V(N)$ is nilpotent group without torsion, then $R(\bar{G})$ is an Ore ring.*

PROOF. We shall prove that all conditions of Proposition 1 are satisfied.

First of all, as $V(N)$ is a fully invariant subgroup of N, we have $V(N) \lhd F$.

Let us suppose now that $V(N) \not\subseteq \Gamma_2(N)$. In this case we obtain from Levy's theorem (see [7], p. 237) that the factorgroup $\bar{N}/\Gamma_2(\bar{N})$ is an abelian group of some finite exponent n; this is impossible, however, because $\bar{N}$ is a nilpotent group without torsion. So, we obtain that $V(N) \subseteq \Gamma_2(N)$.

The group ring $R(\bar{N})$ of a nilpotent group without torsion is, as is known, an Ore ring.[†] Proposition 1 gives now that $R(N)$ is an Ore ring too.

The theorem is proved.

LEMMA 4. *Let G be any group, H_i, $i \in I$, be some system of normal subgroups and $\bigcap_{i \in I} H_i = E$, where E is the unity subgroup. Let $i \leqq j$ if $H_i \supseteq H_j$ and let us suppose that I is a directed set under this order: for any two subgroups H_{i_1} and H_{i_2} there exists a subgroup H_{i_3}, such that $H_{i_1} \supseteq H_{i_3}$, $H_{i_2} \supseteq H_{i_3}$. Then $R(G)$ is a subdirect sum of rings $R(\bar{G}_i)$, where $R(\bar{G}_i)$ is the group ring of the group $\bar{G}_i = G/H_i$.*

PROOF. Let $x = \sum_{k=1}^{n} \lambda_k g_k$ be some element of $R(G)$. Using the condition $\bigcap_{i \in I} H_i = E$, we can for any element $g_k^{-1} g_j$ choose a subgroup $H_{k,j}$ such that

$$g_k^{-1} g_j \notin H_{k,j} \quad 1 \leqq k \leqq n, \quad 1 \leqq j \leqq n, \quad k \neq j. \tag{11}$$

Now let H_{i_0} be a subgroup that satisfies the condition $H_{i_0} \subseteq \bigcap_{k,j} H_{k,j}$.

It follows from (11) that the images $\bar{g}_k$ of the elements g_k, $k = 1, 2, \cdots, n$, under the natural homomorphism $G \xrightarrow{\phi} G/H_{i_0} = \bar{G}_{i_0}$ are all different. The homomorphism $G \to G_{i_0}$ induces an homomorphism $R(G) \to R(\bar{G}_{i_0})$ and the image $\bar{x} = \sum \lambda_k \bar{g}_k$ of x is nonzero in $R(\bar{G}_{i_0})$. We proved that $R(G)$ is approximated by the rings $R(\bar{G}_i)$ and this is equivalent to the conclusion of the lemma.

THEOREM 2. *Let F be a free group, $N \lhd F$ and $V(N) \neq N$ be a fully invariant subgroup of N. If $G = F/N$ is approximated by groups from the class A and $\bar{N} = N/V(N)$ is approximated by nilpotent groups without torsion, then the group ring $R(\bar{F})$ of the group $\bar{F} = F/V(N)$ has no divisors of zero and is embedded in a division ring.*

PROOF. There exists in F a system X of normal subgroups N_α such that $\bigcap_{N_\alpha \in X} N_\alpha = N$ and the factorgroups $G_\alpha = F/N_\alpha$ are groups of class A. If the

[†] The proof follows now from the existence in any finitely generated nilpotent group G without torsion of a series $G = H_k \rhd H_{k-1} \cdots \rhd H_2 \rhd H_1 \rhd H_0 = 1$ where all factors H_{i+1}/H_i are infinite cyclic groups.

system X is not closed under finite intersections we can join to it the finite intersections of the subgroups N and obtain a new system $\bar{X}$; it follows from Lemma 3 and its consequence that if $N_\beta \in \bar{X}$ then $G_\beta = F/N_\beta$ is group of a class A. We will suppose, therefore, that the system X itself is closed under finite intersections.

Let now $\Gamma_k(\bar{N})$ be as above the kth term of the lower central series of $\bar{N}$; $\Gamma_k^*(\bar{N})$ is the isolator of $\Gamma_k(\bar{N})$: $h \in \Gamma_k^*(\bar{N})$ if and only if $h^{n(h)} \in \Gamma_k(\bar{N})$, for some natural $n(h)$. It is well known that $\Gamma_k^*(\bar{N})$ is a fully invariant subgroup of $\bar{N}$.

The following properties of the subgroups $\Gamma_k^*(\bar{N})$, $k = 1, 2, \cdots$, hold.

1) The factorgroup $\bar{N}/\Gamma_k^*(\bar{N})$ is a nilpotent group without torsion of the class k.

This follows from the determination of $\Gamma_k^*(\bar{N})$.

2) $\bigcap_{k=1}^{\infty} \Gamma_k^*(\bar{N}) = E$, where E is the unity subgroup of $\bar{N}$.

In fact, let $1 \neq \bar{g} \in \bar{N}$ and $g \in \bigcap_{k=1} \Gamma_k^*(\bar{N})$. Hence, there exist for any $\Gamma_k(\bar{N})$ such n_k, that $\bar{g}^{n_k} \in \Gamma_{n_k}(\bar{N})$ and this is a contradiction to the approximability by nilpotent groups without torsion.

If now $V_k(N)$ is the inverse image of $\Gamma_k^*(\bar{N})$ under the natural homomorphism $N \rightarrow N/V(N) = \bar{N}$, we have

$$\bigcup_{k=1}^{\infty} V_k(N) = V(N) \tag{12}$$

and

$$V/V_k(N) \cong V/V(N) \Big/ V_k(N)/V(N) \simeq \bar{N}/\Gamma_k^*(\bar{N}).$$

Thus, $V/V_k(N)$ is a nilpotent group without torsion and the subgroup $V_k(N)$ is fully invariant in N.

Let us consider now two possibilities for cardinality of the set of generators of the group F.

1) *F is a free group of infinite rank.*

There exists one-to-one correspondence of the set of fully invariant subgroups of N into the set of fully invariant subgroups of F. We will denote by $V_k(F)$ the verbal subgroup of F which is an image of $V_k(N)$ under this correspondence; $V_k(F)$ is generated by the same words (in F) as $V_k(N)$ (see [7], p. 238). As F_α is a subgroup of F, we obtain also a set $V_k(F_\alpha)$ of verbal subgroups of the group F_α, where $F_\alpha \in X$ and $k = 1, 2, \cdots$.

As the system X is closed under finite intersections it is possible to use Dunwoody's theorem [5] to obtain:

$$\bigcap_{F_\alpha \in X} V_k(F_\alpha) = V_k(\bigcap_{F_\alpha \in X} F_\alpha) = V_k(N), \tag{13}$$

$k = 1, 2, \cdots$.

It follows now immediately from (12) and (13)

$$\bigcap_{k=1}^{\infty} \bigcap_{F_\alpha \in X} V_k(F_\alpha) = \bigcap_{k=1}^{\infty} V_k(N) = V(N). \tag{14}$$

We see that the group $\bar{F} = F/V(N)$ is approximated by groups $\bar{F}_{k,\alpha} = F/V_k(F_\alpha)$. The group $\bar{F}_{k,\alpha}$ is an extension of nilpotent group without torsion $\bar{F}_\alpha = F_\alpha/V_k(F_\alpha)$ by the group $F/V_k(F_\alpha) \big/ F_\alpha/V_k(F_\alpha) \simeq F/F_\alpha \simeq F/F_\alpha = G_\alpha$.

As the group G_α is a group of class A we obtain from Theorem 1 that the group ring $R(\bar{F}_{k,\alpha})$ is an Ore ring.

We shall prove now that the system $V_k(F_\alpha)$, $F_\alpha \in X$, $k = 1, 2, \cdots$, satisfies the conditions of Lemma 4.

Really, if we have two subgroups $V_{k_1}(F_{\alpha_1})$ and $V_{k_2}(F_{\alpha_2})$ then

$$V_{k_1}(F_\alpha) \cap V_{k_2}(F_\alpha) \supseteq V_k(F_{\alpha_1}) \cap V_k(F_{\alpha_2}),$$

where $k = \max(k_1, k_2)$. As X is closed under finite intersections, it follows immediately:

$$V_k(F_{\alpha_1}) \cap V_k(F_{\alpha_2}) \supseteq V_k(F_{\alpha_3}),$$

where $F_{\alpha_3} = F_{\alpha_1} \cap F_{\alpha_2}$.

Therefore

$$V_{k_1}(F_{\alpha_1}) \cap V_{k_2}(F_{\alpha_2}) \supseteq V_k(F_{\alpha_3}),$$

i.e. the conditions of Lemma 4 are fulfilled.

We obtain now from Lemma 4 that $R(\bar{F})$ is approximated by Ore rings $R(\bar{F}_{k,\alpha})$. Therefore $R(\bar{F})$ has no nilpotent elements; as $\bar{F}$ is a group without torsion we obtain (see [10], [4]) that $R(\bar{F})$ has no divisors of zero. However, as known (see [3], [2]), a ring without divisors of zero being a subdirect sum of rings embedded in division rings can be embedded in some division ring itself. Hence, we proved that $R(\bar{F})$ can be embedded in division ring and this completes the consideration of the possibility 1.

2) *F is finitely generated group.*

We shall reduce this case to the first one. Let F_1 be a free group of countable rank; we have $F_1 \simeq F * F_0$. If N_1 is the normal subgroup of F_1 which is generated by F_0 and N, it is easy to check that $F_1/N_1 \simeq F/N \cong G$. Moreover, if $\bar{F}_0$ is the normal closure of F_0 in F_1, then $N \cap \bar{F}_0 = E$ and N_1 is a semidirect product of N and $\bar{F}_0$. Thus, $N_1 \cap F = N\bar{F}_0 \cap F = N$.

As F_1 is a group of countable rank, there exists as above, in the case 1, a verbal subgroup $V(F_1)$ such that $V(F_1)$ and $V(N)$ are generated by the same set of words.

We have now: $F \cap V(N_1) = V(F \cap N_1) = V(N)$ (see [9], chap. 4, 42.41). Thus, we obtained that the group $F/V(N)$ can be embedded isomorphically in the group $F_1/V(N_1)$. It has already been proved that the group ring $R(\bar{F}_1)$ of the group $\bar{F}_1 = F_1/V(N_1)$ can be embedded in a division ring, and, hence, this is true also for the ring $R(\bar{F})$.

The theorem is proved.

COROLLARY 1. *Let F be a free group, $N \triangleleft F$ and $V(N) \neq N$ is a fully invariant subgroup of N. If F/N is a finitely approximated group and $N/V(N)$ is approximated by nilpotent groups without torsion then the group ring $R(\bar{F})$ has no divisors of zero and can be embedded in a division ring.*

COROLLARY 2. *Let F be a free group, $N \triangleleft F$ where F/N is finitely approximated, $N_{n_1, n_2, \cdots, n_k}$ be the term of a polycentral series that corresponds to the sequence of natural numbers $n_1, n_2, \cdots, n_k$. Then the group ring $R(\bar{F})$ of the group $\bar{F} = F/N_{n_1, n_2, \cdots, n_k}$ has no divisors of zero and can be embedded in a division ring.*

The proof follows from Theorem 2 and from the fact that the free polynilpotent group $\bar{N}_{n_1, n_2, \cdots, n_k} = N/N_{n_1, n_2, \cdots, n_k}$ is approximated by nilpotent groups without torsion (see [11]).

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